Section Notes 2

If you notice any errors in this set of lecture notes, please let me know by email. Thanks.

Reminders about Temperature, Energy and Power

To convert between Celsius and Kelvin remember that:

\[ 0 \, ^\circ C + 273 = 273 \, K, \]
\[ 100 \, ^\circ C + 273 = 373 \, K. \]

Energy is a concept essential to science, technology, politics, etc. but it is extremely difficult to give a good definition of what energy is. For our purposes the following definition will be useful:

**Energy**: The ability to do work.

Of course now we must give a rigorous definition for the word work, but already it is intuitively clear that we cannot define energy by its essential essence, only by one of its many potential actions. The reason for this is that energy comes in a large variety of forms.

**Work**: A force acting over some distance.

OK, now at least we have succeeded in defining work to the extent that we understand what a force and a distance is. This is useful because we can apply this definition to assign a measure and a set of units to a quantity of energy. Since work and energy are equivalent we can write the following symbolic statement:

\[ \text{Energy} \rightarrow \text{Work} = \text{Force} \times \text{Distance}. \]

From this we can see that the units of energy are the units of forces times the units of distance:

\[ \text{unit [Energy]} = \left( \frac{\text{kg} \, \text{m}}{\text{sec}^2} \right) \, (\text{m}) = \frac{\text{kg} \, \text{m}^2}{\text{sec}^2} = \text{Joule} \]

So now we have both a unit for measuring energy, the Joule (J), as well as an energy scale: one Joule is the amount of energy expended when one Newton of force is applied over one meter.

The Watt is a measure of the consumption or production of energy. That is why a lightbulb is rated in Watts—it consumes some amount of energy per second.

\[ 1 \, \text{Watt} = 1 \, \text{Joule Second}^{-1} = 1 \, \frac{\text{kg} \, \text{m}^2}{\text{sec}^4}, \]
Electrical power is measured in Watts (W) and Joules (J), which are the International System of Units (SI) units for power and energy, respectively. Non-SI units of energy include the British Thermal Unit (BTU), calorie (cal), and erg. To convert these units to SI units, refer to the appendices of scientific references like *Spherical Cow*. For instance, a quantity of energy given as power (W) times time (s) is a Watt-second, or kilowatt-hour. Since a Watt is defined as 1 Joule per second, 1 Watt-second is equivalent to 1 Joule.

**Electromagnetic Radiation**

Electrons in motion emit electromagnetic radiation—disturbances in the electric and magnetic fields around them that propagate as waves. All waves (including water waves and earthquake waves) have a characteristic wavelength and frequency (see 3).

The wavelength of a wave is the spatial distance from one peak of the wave to the next. It has the dimensions of length and is usually denoted by the Greek letter lambda (\( \lambda \)). The wavelengths of electromagnetic radiation vary from greater than 10 centimeters for radio waves to less than 10^{-9} centimeters for gamma rays. Waves with short wavelength carry more energy than waves with long wavelength. This is why we wear sunscreen (to block ultra-violet radiation) but not 'radio-screen' to protect us from our local radio station.
The frequency, \( f \), of a wave is the number of peaks that pass a fixed point in space over an interval of time. It is usually specified in cycles per second or Hertz. A wave with a frequency of 2 Hertz sends two peaks past a fixed point in space every second. The fundamental unit of frequency is time\(^{-1}\). Frequency is usually specified in units of sec\(^{-1}\) or Hertz (Hz).

In general, the amount and type of radiation emitted by a real object is very difficult to calculate. However, if we make a simplifying assumption about the object it will allow us to use some powerful equations for making predictions. We will assume that the object is a perfect “black-body.” We know from experience walking around barefoot in the summer that black surfaces absorb more light from the sun and thus get hotter. This is related to the black-body assumption. An ideal black body absorbs and radiates with perfect efficiency. The sun is very close to a perfect black-body—but the sun is clearly not black! Actually the sun is black, but it is so hot that it emits visible radiation and appears to be yellow. The Earth is not really a black-body but we will often assume that it is for simplicity. We can then make adjustments to account for the error this introduces.

The relationship between wavelength and frequency is simple and important to remember:

\[
\lambda = \frac{c}{f},
\]

with \( c \) being the speed of the wave (if you forget, just think of the units of \( \lambda \), \( c \) and \( f \)). The speed of electromagnetic radiation in a vacuum is 299,792.458 kilometers per second. This is usually rounded off to \( 3 \times 10^8 \) meters/second. In air the speed is only slightly lower. Thus we can rewrite the above formula as

\[
\lambda = \frac{3 \times 10^8}{f},
\]

where \( f \) is given in seconds and \( \lambda \) is given in meters. This formula tells us that knowing the wavelength and frequency of an electromagnetic wave is redundant. Given one we can calculate the other with no more information.

As we said before, the wavelength (and thus frequency) of radiation emitted by a body is related to the absolute temperature of that body. However, since temperature is a measure of the average energy of vibration of particles in a body, it can only predict the most intense wavelength of light emitted. The intensity of radiation emitted by an object as a function of wavelength or frequency \((I(\lambda) \text{ or } I(f))\) is known as a spectrum. The relationship between absolute temperature and the wavelength with highest intensity is Wein’s Law (exactly true only for a black-body):

\[
\lambda = \frac{2.898}{T}.
\]

Here \( \lambda \) is in micrometers and \( T \) is in Kelvin. The full spectrum of a body emitting radiation is spread out around the peak wavelength, as seen in figure 4.
The radiative intensity per unit area of a black-body at temperature $T$ can also be predicted. This intensity represents the sum of waves emitted at all wavelengths. It is given by the Stefan-Boltzmann law:

$$I = \sigma T^4,$$

Where

$$\sigma = 5.669 \times 10^{-8} \frac{J}{m^2 K^4 \text{sec}}.$$

Again, this formula is only exactly true for a black-body. It is a good approximation for things that are close to black. Notice how the units of sigma, when multiplied by $K^4$, give the units Watts/m². This tells you that radiative intensity is equivalent to power per area. One Watt per square meter is one Joule of energy distributed over a window of area 1 m² and a time of 1 second.

If we wanted to find the total power (the total power output) of the sun we would multiply the intensity of the sun by the area of the sun’s emitting surface. Hey, let’s do that:

$$\text{Power} = \text{Intensity} \times \text{Area} = (\sigma T^4_{\text{sun}})(4\pi R^2_{\text{sun}})$$

$$= 5.669 \times 10^{-8} \frac{J}{m^2 K^4 \text{sec}} \times 5800^4 K^4 \times 4\pi \times (6.96 \times 10^8)^2 \text{ m}^2$$

$$= 3.9 \times 10^{26} \text{ W!!}$$

That is a lot of power! The largest power plant on Earth delivers only $8.6 \times 10^9$ Watts\textsuperscript{1}.

\textsuperscript{1}http://www.solar.coppe.ufrj.br/itaipu.html
The Earth’s energy balance

Because energy cannot be created or destroyed, we know that the power that leaves the sun must be evenly distributed over a spherical “window” around the sun. To find the flux of the sun’s power through a window with a radius equal to $R_{Sun-Earth}$ we can write:

$$F_{Earth} = \frac{\text{Total power of Sun}}{\text{Area, sphere of radius } R_{Sun-Earth}} = \frac{3.9 \times 10^{26} \text{ W}}{4\pi R_{Sun-Earth}^2} \equiv S_0 = 1368 \text{ W m}^{-2}.$$ 

This number, $S_0 = 1368 \text{ W m}^{-2}$ is known as the solar constant. It is the intensity of light from the Sun incident on the Earth.

The Earth’s average temperature, over a short enough time scale, is roughly constant. Clearly we must average over enough time to eliminate seasonal and yearly fluctuations but not long enough to go into an ice age. This means that on average, at the top of the atmosphere, incoming solar radiation (short wavelength) is balanced by outgoing thermal (long wavelength) radiation. Thus we say that the Earth is in radiative balance with the Sun. If this were not the case the Earth’s average temperature would begin to change very rapidly! The average temperature of the Earth is about 15 °C or 288 K. Because of the strong constraint of radiative balance, we can model the average temperature of the Earth in a fairly simple way.

We will model the Earth as a sphere in space with the assumption that the average temperature of the sphere is constant. The meaning of radiative balance is:

$$\text{Incoming solar radiation} = \text{Outgoing thermal radiation},$$

$$S_0 \times \text{(absorbing area)} = \sigma T^4 \times \text{(emitting area)}.$$ 

Using this simple equation and the geometry of a sphere we can predict the temperature of the Earth.

Consider the three objects shown in figure 3. They are each orbiting the Sun at the same distance as does the Earth. They are each perfect black-bodies so they absorb the solar radiation, $S_0$, over their areas and reradiate according to their respective temperatures back into space. The plate shown in figure 3a is insulated on one side. Since it is oriented perpendicular to the solar radiation it absorbs $S_0$ Watts over its area. By assuming radiative balance for the plate we can calculate its temperature:

$$T = \left( \frac{S_0}{\sigma} \right)^{1/4} = 394 \text{ K} = 121 \degree \text{C}.$$ 

For the plate in figure 3b we have almost the same thing. The only difference is that the plate emits radiation over both sides. Thus,

$$T = \left( \frac{S_0}{2\sigma} \right)^{1/4} = 331 \text{ K} = 58 \degree \text{C}.$$
Finally, we consider the sphere (Earth) shown in figure 3c. The shadow of the sphere is shown on the “screen” hanging in space behind the Earth. This tells us that the Earth absorbs solar radiation over its cross sectional area, $\pi R_{Earth}^2$. However, since the Earth rotates it is approximately uniform in temperature, and thus it radiates over its entire surface area, $4\pi R_{Earth}^2$. Thus, solving for the Earth’s temperature we get:

$$T = \left( \frac{S_0}{4\sigma} \right)^{1/4} = 279 \text{ K} = 6 \, ^\circ\text{C}.$$ 

Not very far from 288 K! Unfortunately, this model lacks an important feature that will be discussed in the next section.

**Planetary albedo**

Here we must face a rather important fact: the Earth is not a black-body. The problem is that if we throw out this assumption we lose the ability to model the Earth in such a simple and powerful way because the Stefan-Boltzmann law will no longer apply. So instead of tossing our assumption, we add a correction to the model to account for the fact that the Earth actually reflects a good portion of the light that is incident upon it. In fact, based on satellite measurements, we know that the Earth reflects about 31% of incident solar energy. That energy bounces off the light colored parts of the Earth: clouds, ice sheets, snow, the desert, etc. This is called the planetary albedo. We can define the unit-less parameter $A = 0.31$ that tells us what fraction of incident solar radiation is reflected (the amount absorbed is thus $1 - A$). With this correction the Earth absorbs $\pi R_{Earth}^2(1 - A)S_0$ Watts of power. Using this new absorption to calculate the temperature of the Earth we get

$$T = \left[ \frac{S_0(1 - A)}{4\sigma} \right]^{1/4} = 254 \text{ K} = -19 \, ^\circ\text{C}.$$ 

Although a it was calculated using a better model of the radiation balance, this number is quite a bit further from the Earth’s average temperature than our last estimate. Actually, what this number represents is the Earth’s *effective temperature*, the average temperature at which it radiates to space. Why is this different from 288 K? Well the answer lies in the fact that we have neglected to consider the Earth’s atmosphere in this model. We take up that issue in the next section.

**Atmospheric absorption & the greenhouse effect**

We know that the atmosphere absorbs radiation in many frequency “bands” because of the presence of greenhouse gases such as water vapor, carbon dioxide, ozone, methane, etc. This will clearly affect the heat balance of the Earth because not all solar radiation will strike the surface and not all
Figure 3: Three objects in space with the same distance from the Sun as the Earth. (a) An absorbing plate 1 m square insulated on the back. It emits radiation from one side only. (b) An absorbing plate 1 m square. It absorbs radiation over 1 square meter and emits it over 2, front and back. (c) A sphere (Earth) that absorbs radiation over its cross-sectional area ($\pi R^2$) and emits radiation over its entire surface ($4\pi R^2$).
Figure 4: A schematic diagram of the one layer atmosphere model. Note that the greenhouse layer is transparent to incident solar radiation (high-$f$) but opaque to outgoing thermal radiation (low-$f$). The two radiative balance equations are formulated by requiring that downgoing radiation is equal to upgoing radiation at the top of the atmosphere and just above the surface.

Radiation emitted by the surface will go directly to space. The overall effect of the atmosphere as an absorber and reemitter of radiation is known as the greenhouse effect.

To improve our understanding of the Earth’s average temperature we’d like to incorporate the greenhouse effect into our model, but without adding too much complexity. We can do this by considering a one layer atmosphere as shown in 4. Solar radiation not reflected by the albedo passes through the greenhouse layer (we are assuming no absorption of high frequency radiation) and is absorbed at the surface of the Earth. This energy is reradiated according to the Stefan-Boltzmann law. However, the greenhouse layer of the atmosphere contains greenhouse gases that strongly absorb all the low frequency radiation emitted by the surface of the Earth. The this layer then emits radiation according to the Stefan-Boltzmann law upwards towards space and downwards toward Earth (see fig. 4). To maintain radiative balance we know that at the top of the atmosphere, the radiation emitted from the greenhouse layer must balance the incoming (non-reflected) solar energy:

$$\sigma T_{upper}^4 = \frac{S_0(1 - A)}{4}.$$  \hspace{1cm} (1)

At the surface, radiation is arriving directly from the Sun as well as from the greenhouse layer of
the atmosphere above. Radiative balance at the surface states

\[ \sigma T_{sfc}^4 = \sigma T_{upper}^4 + \frac{S_0 (1 - A)}{4}. \]  

(2)

To solve for the surface temperature we must substitute equation 1 into equation 2 and solve for \( T_{sfc} \) with the result

\[ T_{sfc} = \left[ \frac{2S_0 (1 - A)}{4\sigma} \right]^{1/4} = 302 \text{ K} = 29 \text{ °C}. \]

Now we are getting closer. Our assumption that all solar radiation not reflected by the albedo of the Earth reaches the surface is probably not such a good one. Furthermore it is a gross oversimplification to assume that the atmosphere absorbs and radiates at one temperature \( T_{upper} \). In fact, to give a rough approximation of the radiation structure of the atmosphere we would need to include five different absorbing and radiating layers. Nonetheless, it is quite amazing that such a simple model of the Earth’s climate can give us such a good prediction of the average temperature at the surface. This same technique can be applied to other planets in the solar system with similar success.

The difference between the effective temperature of a planet and the actual average surface temperature gives a good measure of the strength of the greenhouse effect of the atmosphere of that planet. For Earth this difference is about 34 °C. That’s quite a lot when you consider that the difference between present global average temperature and that during an ice age is only about 10 degrees. Clearly we depend on Earth’s greenhouse effect to moderate the climate. Increasing the concentration of greenhouse gases in the atmosphere could thus be very dangerous.

We know by various means that the concentration of CO\(_2\) in the atmosphere has been correlated with temperature over the past 100 million years. During the cretaceous the Earth was very warm and CO\(_2\) levels were much higher. As we are learning now, drastically increasing the amount of greenhouse gases in the atmosphere does not have an comensurately drastic immediate effect on global average temperature. We must keep in mind that the climate system is extremely complex and may be prone to change in ways that do not behave linearly in terms of global average quantities.

Massive volcanic eruptions, large meteorite impacts and nuclear explosions have the potential to cause enormous change of climate by affecting the albedo. An atmosphere that is thick with fine dust will tend to reflect back more of the incident solar radiation and cool the Earth. This can obviously happen on a very short, almost immediate time scale.

Our simple one layer model of the atmosphere is also instructive of how more complicated climate models (known as general circulation models or GCMs) work. Instead of dividing up the atmosphere into a single shell, GCMs divide the atmosphere into a large set of stacked shell that are then divided further into small boxes. The radiative balance equations are then applied to each
box (this is carried out on massive computers). Of course there are other equations that must be solved on the “grid” to calculate things such as wind and moisture.